City Guarding with Cameras of Bounded Field of View

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Abstract

We study two problems related to the city guarding and the art gallery problems.

1. Given a city with k rectangular buildings, we prove that $3k+1$ cameras of $180°$ field of view are always sufficient to guard the free space (the ground, walls, roofs, and the sky). This answers a conjecture of Daescu and Malik (CCCG, 2020).

2. Given k orthogonally convex polygons of total m vertices in the plane, we prove that $\frac{m}{2}+k+1$ cameras of $180°$ field of view are always sufficient to guard the free space (avoiding all the polygons). This answers another conjecture of Daescu and Malik (Theoretical Computer Science, 2021).

Both upper bounds are tight in the sense that there are input instances that require these many cameras. Our proofs are constructive and suggest simple polynomial-time algorithms for placing these many cameras.

1 Introduction

Fixed cameras are common devices that are being used to monitor streets and buildings in cities. These cameras usually monitor the ground and walls. Due to an increasing use of drones and other flying objects, monitoring the entire space (including the ground, walls, roofs, and sky) is becoming crucial. The problem of monitoring the entire space with minimum number of cameras is usually referred to as the *city guarding* problem in computational geometry.

To the best of our knowledge the problems related to guarding cities were first introduced by Bao et. al [2]. They introduced three different versions of the problem where the goal is to guard (1) only the roofs of the buildings, (2) the walls of the buildings and the ground, and (3) the roofs, walls, and the ground. This latter version is called "city guarding".

In the city guarding problem we should take into account many factors such as the city's layout, buildings' orientation, and the cameras' field of view. These factors usually led to different variations of the city guarding problem.

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In this paper we study a version of the city guarding problem that is introduced by Daescu and Malik [5]: *Given k pairwise disjoint rectangular-base buildings, find a minimum number of cameras that guard the city such that (i) each camera is a half-sphere with* $180°$ *field of view and infinite range, and (ii) each camera is placed at a corner on top of the roof of a building in a direction orthogonal to a wall.*

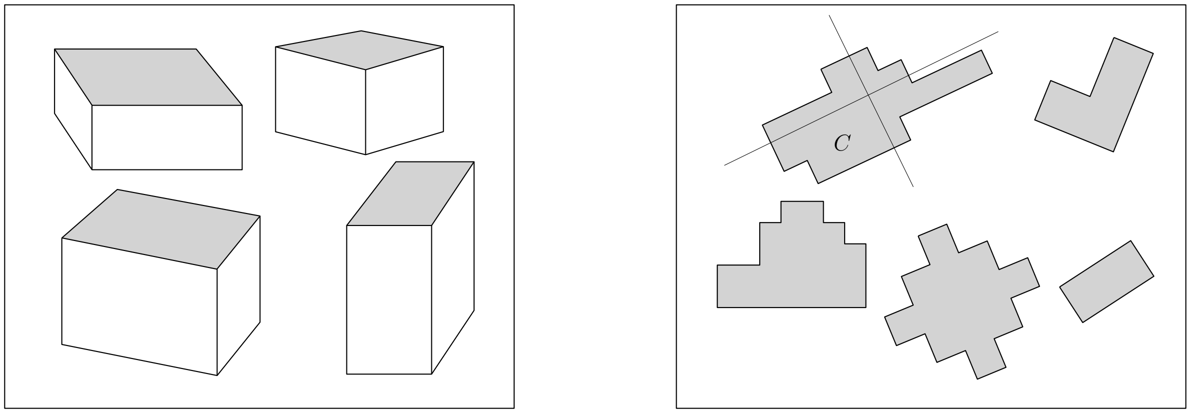
According to Bao et al. [2] the city guarding problem can be interpreted as a 2.5-dimensional version of the well-studied art gallery problem. In the standard art gallery problem we are given a simple polygon and the goal is to place the minimum number of guards/cameras to cover the entire polygon [8]. In other words, each point of the polygon is visible by some guard. A point p is said to be *visible* by a guard g if the line segment pg lies inside the polygon. The art gallery problem and its variations have been well-studied in recent years [12]. The variations usually enforce constraints on the shape of the polygon, the existence of holes, the shape of holes, the orientation of holes, locations of guards, guards' field and range of vision, to name a few. The city guarding problem has the same flavor as the art gallery problem with rectangular holes.

The city guarding also has the same flavor as a free-space illuminating problem, studied by Blanco et al. [3], in which the input consists of pairwise disjoint rectangles in the plane and the goal is to place minimum number of lights at the corners of the rectangles to light up the free space (the entire plane minus the the rectangles).

2 Related Works and Results

In this section we focus only on results that are directly related to the city guarding problem. There is a rich literature for the art gallery problem for which we refer the reader to [1, 3, 4, 7, 8, 9, 10, 11].

Bao et al. [2] studied the city guarding problem for k rectangular-base buildings that are orthogonal (to the xy-axis) and for cameras with $360°$ field of view. Recall that the cameras should be placed at top corners of buildings. They showed that $\lfloor\frac{2(k-1)}{3}\rfloor+1$ guards are always sufficient and sometimes necessary to guard the roofs. They also showed that $k+\lfloor\frac{k}{4}\rfloor+1$ guards are sufficient to guard walls and ground. For the city guarding (roofs, walls, the ground) they showed the sufficiency of



(a) (b)

Figure 1: (a) A city with rectangular buildings. (b) Orthogonally convex polygons.

$k + \lfloor \frac{k}{2} \rfloor + 1$ guards.

Recently, Daescu and Malik [5] studied the city guarding problem for cameras with 180° field of view. They proved that $2k + \lfloor \frac{k}{4} \rfloor + 4$ cameras are sufficient to guard axis-aligned buildings. For arbitrary oriented buildings they gave an example that requires $3k + 1$ cameras for any $k \geq 1$. They conjectured that $3k + 1$ cameras are also sufficient. See Figure 1(a) for an example of arbitrary-oriented rectangular buildings.

In a companion paper, Daescu and Malik [6] studied another problem of the same flavor; guard free space formed by orthogonally convex polygons. Given k pairwise disjoint orthogonally convex polygons with total m vertices, the goal is to place cameras of 180° field of view to guard the free space and the boundaries of the polygons (cameras should be placed at corners of polygons and orthogonal to its sides). An *orthogonal polygon* is a polygon whose edges are orthogonal to each other (not necessarily orthogonal to the xy-axis). An orthogonal polygon is *orthogonally convex* if its intersection with any line orthogonal to its edges is either empty or a single line segment; see for example polygon C in Figure 1(b). Daescu and Malik show that for axis-aligned polygons $\frac{m}{2} + \lfloor \frac{k}{4} \rfloor + 4$ cameras are always sufficient and for arbitrary-oriented polygons $\frac{m}{2} + k + 1$ cameras are sometimes necessary for any $k \geq 1$ and any valid m. They conjectured that $\frac{m}{2} + k + 1$ cameras are also sufficient. See Figure 1(b) for an example of arbitrary-oriented orthogonally convex polygons.

2.1 Our Contributions

We prove both conjectures of Daescu and Malik [5, 6] that $3k + 1$ cameras are sufficient to guard arbitrary-oriented rectangular buildings, and $\frac{m}{2} + k + 1$ cameras are sufficient to guard arbitrary-oriented orthogonally convex polygons. Our proofs are constructive and suggest polynomial-time algorithms for finding these many guards. The two proofs share some similarities in the sense that both partition the free space into convex regions and then provide an upper bound for the number of these regions. We explain our proof for rectangular buildings first as it is easier to explain. Then we give a short description of how to generalize it for monotone orthogonal polygons.

3 City Guarding

In this section we present our algorithm for the city guarding problem. The following lemma, borrowed from [5], implies that to guard the entire space it suffices to guard roofs, walls, and the ground. Therefore in the algorithm we focus on guarding roofs, walls, and the ground.

Lemma 1 (Daescu and Malik [5]) *If in a city the roofs, walls, and the ground are guarded by a set of cameras, then every point in the aerial space of the city is visible by a camera.*

Recall that the city consists of k arbitrary-oriented buildings with rectangular basis, and that the cameras have 180° field of view and should be placed at corners on top of the roofs orthogonal to a wall. (We clarify that a camera could be placed in such a way that it sees the roof of the building, as in Figure 3.)

Daescu and Malik [5] gave an example which requires $3k + 1$ cameras. This example is given in Figure 2. Each building B_{i+1} is higher than the building B_i. They conjectured that the bound $3k + 1$ is tight.

We show how to guard the city with at most $3k + 1$ cameras, and thus proving the conjecture of [5]. We project the buildings onto the plane to obtain rectangles (in dimension 2). Then we guard the the rectangles (representing roofs), their sides (representing walls), and the space between them (representing the ground).

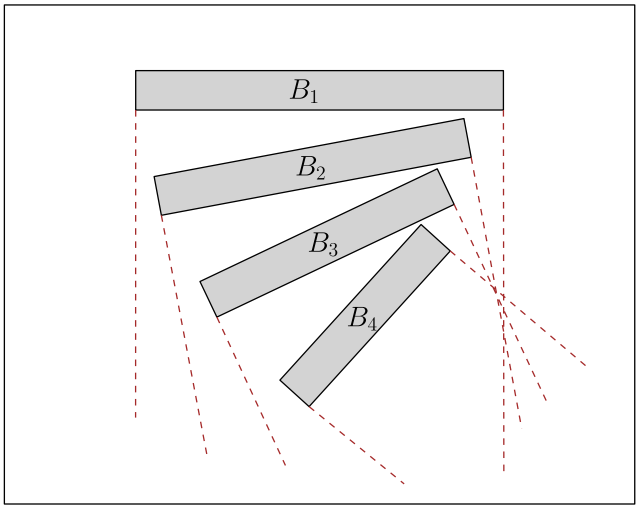


Figure 2: A city with $k = 4$ buildings that needs $3k + 1$ guards; borrowed from [5].

By Lemma 1, this would give a guarding of the city in dimension 3.

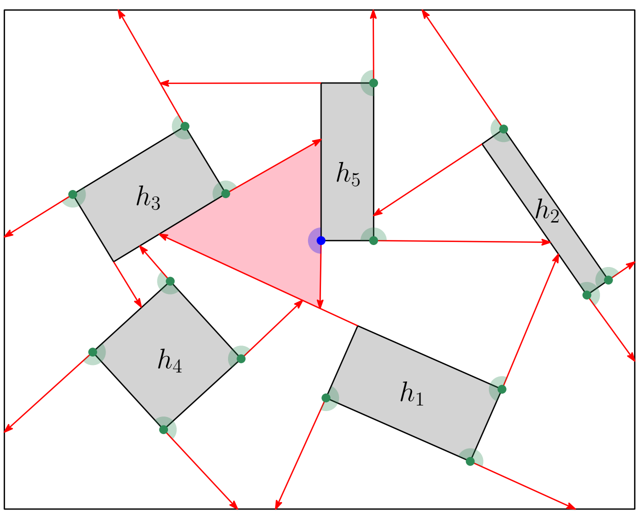


Figure 3: A city with $k = 5$ buildings. The sides are extended in order h_1, h_2, h_3, h_4, h_5. The pink area is a bad region. The green marks are corner guards and the blue mark is an edge guard.

We start by projecting the buildings vertically into the plane; this is a typical first step for problems of this type, see e.g. [2, 5]. Thus we obtain k pairwise disjoint rectangles in the plane. We may assume without loss of generality that the k rectangles lie in a bigger rectangle called P. One can think of P as a polygon and of rectangles as holes. Thus after this projection, each building becomes a hole in P and each wall becomes a side of some hole. One can think of this as an instance of the art gallery problem consisting of a polygon with rectangular holes.

Our next step is to guard P by cameras with 180° field of view. This would give (after lifting the rectangles back to their original height) a guarding of walls and the ground. As we will see later, our placement of cameras would guard the roofs as well.

Let $h_1, h_2, \ldots, h_k$ denote the rectangular holes ordered arbitrarily. For each h_i in this order, we extend the sides of h_i in counterclockwise direction and stop as soon as reaching another hole, an extension of a previous side, or the boundary of P; see Figure 3. Each extension is essentially a *directed line segment* whose initial point is a hole corner. These extensions partition P into some regions that we denoted $R_1, R_2, \ldots$; notice that we exclude the holes.

Lemma 2 *Each region R_i is convex.*

Proof. The region R_i is an intersection of a set of quadrants (which are convex). Each quadrant is defined by extensions of two adjacent sides of the same hole. Since the intersection of any set of convex objects is known to be convex, the region R_i is convex. □

Lemma 3 *The number of regions $R_1, R_2, \ldots$ is $3k+1$.*

Proof. We define a plane graph $G = (V, E)$ as follows. The vertex set V consists of the corners of the holes and the intersection points of the extended sides. We refer to them by *corner* and *intersection* vertices, respectively. The edges in E are formed by the sides of the holes, the extensions of sides, and the boundary of P.

We claim that each vertex of G has degree 3, and thus G is 3-regular. Each corner vertex is incident to two sides of a hole and an extension, thus has degree 3. Each intersection vertex is incident to an extension and two segments obtained from the intersected segment, and thus has degree 3. Degenerate cases are rather easy to handle, for example if two extensions hit a segment at the same point p, then we treat p as two vertices of degree 3 instead of one vertex of degree 4.

The number of corner vertices is $4k$. Each extension (of a side of a hole) defines an intersection vertex. Thus the number of intersection vertices is the same as the total number of sides of holes, which is $4k$. Therefore $|V| = 8k$. Since the sum of the vertex degrees in any graph is twice the number of edges and G is 3-regular, we have the following equality,

$$2|E| = 3|V|.$$

Therefore,

$$|E| = \frac{3|V|}{2} = \frac{3 \cdot 8k}{2} = 12k.$$

Let F be the set of faces of G, which includes the holes, the outerface (exterior of P), and the regions

$R_1, R_2, \ldots$. Using Euler's formula for connected planar graphs, we have

$$|F| = |E| - |V| + 2 = 12k - 8k + 2 = 4k + 2.$$

Excluding the outerface and the k holes, the number of regions $R_1, R_2, \ldots$ is $3k + 1$. □

Lemma 4 *Each region R_i contains a corner of a hole on its boundary.*

Proof. Recall the extensions of $h_1, \ldots, h_k$ in this order. Observe that the boundary of R_i contains (parts of) some extensions. Consider the last extension that was added to the boundary of R_i, or say, closes the region R_i. The entire directed line segment that defines this extension is part of the boundary of R_i. The initial point of this directed line segment is a corner of a hole. □

By Lemma 4 each region R_i has a hole corner on its boundary. If the boundary of R_i has a 90° angle at some corner, then we call it a *good region*, and otherwise a *bad region*; see Figure 3.

Camera Placement: Take any region R_i. If R_i is a bad region then let c be an arbitrary corner on the boundary of R_i. We place a camera at c facing towards the interior of R_i and perpendicular to the boundary segment of R_i containing c. We call this camera an *edge guard*—it lies on an edge of R_i. If R_i is a good region then let c be the lowest (i.e. with the smallest y-coordinate) corner at which the boundary of R_i has angle 90°. We place a camera at c facing towards the interior of R_i and perpendicular to the clockwise boundary segment at c (which is the extension at c). We call this camera a *corner guard*—it lies on a corner of R_i.

Since R_i is convex (by Lemma 2) the camera that is placed on the boundary of R_i covers the entire interior of R_i. Since we place exactly one camera for each region R_i, (i) all regions $R_1, R_2, \ldots$ are guarded, and (ii) the number of cameras is equal to the number of regions R_i which is $3k+1$ by Lemma 3. Therefore we have guarded the polygon P by $3k + 1$ guards. As discussed earlier, this gives a guarding of walls and the ground in the city.

We claim that our camera placement, also guards the roofs. Observe that for each hole h it holds that one of its corners is the lowest corner of angle 90° on the boundary of some good region R_i. Notice that such a lowest corner of R_i is uniquely defined by h. The camera that is placed at that corner (perpendicular to the extended side), guards the roof of h. The following theorem summarizes our result of this section.

Theorem 5 *Given k arbitrary-oriented rectangular-base buildings, we can guard the entire space (the ground, walls, roofs, and the sky) with at most $3k + 1$ cameras of 180° field of view that are placed at top corners of buildings orthogonal to a wall. The bound $3k+1$ is the best achievable.*

4 Guarding Orthogonally Convex Polygons

In this section we present our algorithm for guarding the free space formed by orthogonally convex polygons. Recall that the scene consists of k arbitrary-oriented orthogonally convex polygons, and that the cameras have 180° field of view and should be placed on corners of polygons orthogonal to a side. We may assume without loss of generality that the k polygons lie in a rectangular polygon called P. The free space, that we need to guard, is the interior of P minus the k given polygons.

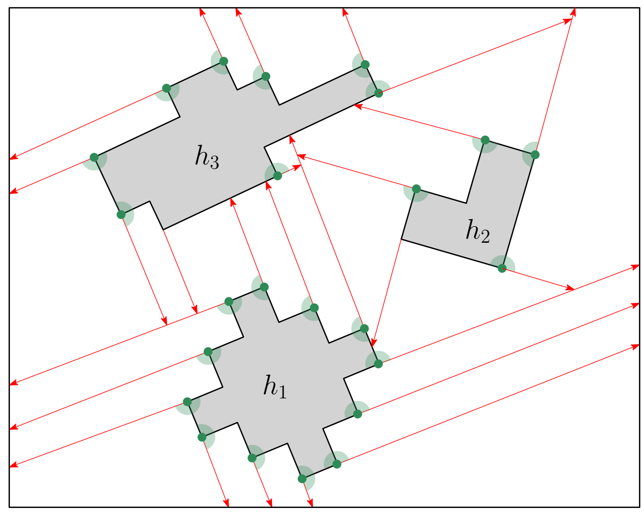


Figure 4: Three orthogonally convex polygons in the plane. The green marks are corner guards.

Similar to our algorithm for the city guarding in previous section we extend the sides of the polygons to partition the free space into convex region and then use one camera for each region. Let $h_1, h_2, \ldots, h_k$ denote the polygons in an arbitrary order. For each h_i in this order, we extend the sides of h_i in counterclockwise direction and stop as soon as reaching another polygon, an extension of a previous side, or the boundary of P. We only extend the sides whose extensions do not intersect the interior of h_i; see Figure 4. Thus we extend one side for every convex corner of a polygon. These extensions partition the free space into some regions that we denoted $R_1, R_2, \ldots$.

By an argument similar to that of Lemma 2 we can show that each R_i is convex.

By an argument similar to that of Lemma 3 we can show that the number of regions R_i is $\frac{m}{2} + k + 1$. We define a 3-regular plane graph $G = (V, E)$ as before. Among all corners, we only introduce vertices for convex ones. By a simple counting argument one can show that the total number of convex corners is $c = \frac{m}{2} + 2k$; see also [6]. Thus the number of vertices of G is $2c$, one vertex for each convex corner and one vertex for its extension. Thus $|V| = 2c = m + 4k$. Since the graph is 3-regular, the total degree is $3|V| = 3m + 12k$, which is equal to $2|E|$. Hence $|E| = \frac{3m}{2} + 6k$. Thus, for the number of faces we get

$$|F| = \left(\frac{3m}{2} + 6k\right) - (m + 4k) + 2 = \frac{m}{2} + 2k + 2.$$

Excluding the outerface and the k holes, the number of regions R_i is $\frac{m}{2}+k+1$. Similar to Lemma 4 we can show that each R_i has a corner on its boundary. We classify the regions by *good* and *bad* and then place cameras on the corners (one camera for each R_i) similar to our placement in the previous section. This would guard the free space with $\frac{m}{2} + k + 1$ cameras. The following theorem summarizes our result in this section.

Theorem 6 *Given k pairwise disjoint arbitrary-oriented orthogonally convex polygons of total m vertices in the plane, we can guard the entire free space with at most $\frac{m}{2} + k + 1$ cameras of $180°$ field of view that are placed at the corners of the polygons orthogonal to a side. The bound $\frac{m}{2} + k + 1$ is the best achievable.*

Remark. It is easily seen that the algorithm of this section can be generalized to guard cities with buildings that have orthogonally convex bases. In fact, the npcity guarding in the previous section is a special case of this problem where $m = 4k$.

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